A Combinatorial Problem on Finite Abelian Groups

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In this paper the following theorem is proved. Let $G$ be a finite Abelian group of order $n$. Then, $n + D(G) - 1$ is the least integer $m$ with the property that for any sequence of $m$ elements $a_1, ..., a_n$ in $G$, 0 can be written in the form $0 = a_1 + \cdots + a_i$, with $1 \leq i_1 < \cdots < i_n \leq m$, where $D(G)$ is the Davenport’s constant on $G$, i.e., the least integer $d$ with the property that for any sequence of $d$ elements in $G$, there exists a nonempty subsequence that the sum of whose elements is 0.

integer \( d \) with the property that for any sequence \( T \) of \( d \) elements in \( G \),
\[ 0 \in \sum (T). \]

For a sequence \( S = (a_1, ..., a_k) \) of elements in an Abelian group \( G \) and any element \( b \in G \), we use \( b + S \) to denote the sequence \((b + a_1, ..., b + a_k)\). We shall obtain Theorem 1 from the following result.

**Theorem 2.** Let \( G \) be a finite Abelian group of order \( n \), \( k \) a positive integer, and \( S = (a_1, ..., a_{n+k}) \) a sequence of \( n+k \) elements in \( G \). Suppose that for every \( b \in G \), and every \((k+1)\)-terms subsequence \( T \) of \( b+S \), \[ 0 \in \sum (T). \] Then, \( 0 \in \sum_n (S) \).

For any sequence \( S \) of elements in an Abelian group \( G \) and any \( b \in G \), we use \( S(b) \) to denote the number of the times that \( b \) occurs in \( S \). By \( h(S) \) we denote the integer \( \max \{ S(b) \mid b \in G \} \).

Theorem 2 is an easy consequence of the following result.

**Theorem 3.** Let \( G \) be a finite Abelian group of order \( n \), \( k \) a positive integer, and \( S \) a sequence of \( n+k \) elements in \( G \). Suppose \( b \) is an element in \( G \) with \( S(b) = h(S) \), and suppose that for every \((k+1)\)-terms subsequence \( T \) of \( -b + S \), \[ 0 \in \sum (T). \] Then, \( 0 \in \sum_n (S) \).

Clearly, Theorem 2 follows from Theorem 3.

**Proof of Theorem 1.** It follows from Theorem 2 that
\[ r(G) \leq n + D(G) - 1. \]

To prove \( r(G) > n + D(G) - 2 \), we consider the following example:

Let \( T = (a_1, ..., a_{D(G)-1}) \) be a sequence of \( D(G)-1 \) elements in \( G \) with \( 0 \notin \sum (T) \), and put
\[ S = (a_1, ..., a_{D(G)-1}, 0, ..., 0). \]

It is easy to see that \( 0 \notin \sum_n (S) \) and \( |S| = n + D(G) - 2 \), this implies that \( r(G) > n + D(G) - 2 \). So we complete the proof.

In [4], Olson proved that if \( p \) is a prime, then \( D(\bigoplus_{i=1}^k Z_{p^i}) = 1 + \sum_{i=1}^k (p^i-1) \); in [5], Olson proved that \( D(Z_{n_1} \bigoplus Z_{n_2}) = n_1 + n_2 - 1 \), where \( n_1 \mid n_2 \). Combining these results and Theorem 1 we have
Corollary 1. (i) If $p$ is a prime, then $r(\bigoplus_{i=1}^{k} \mathbb{Z}/p^i) = p^{\sum_{i=1}^{k} \epsilon_i} + \sum_{i=1}^{k-1} (p^i - 1)$.

(ii) If $n_1 | n_2$, then $r(\mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2) = n_1n_2 + n_1 + n_2 - 2$.

Eggleton and Erdős [2] proved that if $G$ is a noncyclic Abelian group of order $n$, then $D(G) \leq n/2 + 1$, therefore, by theorem 1 we have

Corollary 2. If $G$ is a noncyclic Abelian group of order $n$, then $r(G) \leq 3n/2$.

3

In this section we shall prove Theorem 3, and to do this we need some preliminaries.

Lemma ([1]). Let $S$ be a sequence of $n$ elements in an Abelian group $G$ of order $n$, and $h = h(S)$. Then, $0 \in \sum_{i=1}^{h} (S)$.

Let $S_1, ..., S_t$ be disjoint subsequences of a sequence $S$, deleting the terms of $S_1$ from $S$ we get a subsequence and denote it by $S - S_1$, and by induction we define $S - S_1 - \cdots - S_i$, to be the subsequence $(S - S_1 - \cdots - S_{i-1}) - S_i$, for $i = 2, ..., t$.

For a sequence $U = (c_1, ..., c_t)$ of elements in an Abelian group, we use $\sum U$ to denote the sum $\sum_{i=1}^{t} c_i$.

Proof of Theorem 3. We may assume that $h \leq n - 1$ for the theorem is clearly true when $h \geq n$.

We may assume that $b = 0$ (otherwise we consider $-b + S$ instead of $S$). Clearly, one can rearrange $S$ to the type

$$S = (a_1, ..., a_{n+k-h}, 0, ..., 0)$$

with all $a_i \neq 0$.

Let $W$ be the largest subsequence (in size) of $(a_1, ..., a_{n+k-h})$ with $\sum W = 0$. It follows from the hypothesis of the theorem that $n + k - h - |W| \leq k$, therefore

$$|W| \geq n - h.$$ 

If $|W| \leq n$, then $0$ can be written in the form

$$0 = 0 + \cdots + 0 + \sum_{n-|W|} W,$$

this shows that $0 \in \sum_{n} (S)$. 

102 W. D. GAO
If $|W| \geq n + 1$, repeatedly applying Lemma to $W$ one can find a system of disjoint subsequences $W_i$ of $W$ such that
\[
\sum W_j = 0 \quad \text{and} \quad 1 \leq |W_j| \leq h \quad \text{for} \quad i = 1, \ldots, u
\]
and such that
\[
|W - W_1 - \cdots - W_u| \leq n - 1.
\]
Let $v$ be the least integer $t$ with the property that $|W - W_1 - \cdots - W_v| \leq n - 1$. Then
\[
|W - W_1 - \cdots - W_v| \leq n - 1
\]
and
\[
|W - W_1 - \cdots - W_{v-1}| \geq n,
\]
these imply that
\[
n - 1 \geq |W - W_1 - \cdots - W_v| \geq n - |W_v| \geq n - h.
\]
Put $W_0 = W - W_1 - \cdots - W_v$. Then $n - 1 \geq |W_0| \geq n - h$, and
\[
0 = 0 + \cdots + 0 + \sum_{\alpha - |W_0|} W_0.
\]
This shows that $0 \in \sum_n (S)$ and completes the proof.

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References